Nafta-Gaz 2025, no. 7, pp. 482–489, DOI: 10.18668/NG.2025.07.07

Numerical solutions of elastoplastic problems with application to underground mining stability

Numeryczne rozwiązania zagadnień sprężysto-plastycznych z zastosowaniem w analizie stabilności wyrobisk podziemnych

Rafail K. Mehtiyev, Yusif A. Tanriverdiyev

Azerbaijan State Oil and Industry University

ABSTRACT: A limited area of plastically deformed rocks forms around underground workings due to a specific combination of the strength of the main rock mass, its structure, and burial depth. The size of this zone and the magnitude of displacements along the excavation boundary determine the stability of the excavation. Analytical solutions of elastic-plastic problems are typically limited to simplified models of the medium (solid, isotropic, homogeneous) and excavation shape (circular). Mathematical modeling of elastoplastic deformation in a structurally heterogeneous rock mass weakened by complex-shaped underground workings is performed solely using numerical methods, such as the finite element method (FEM). In this context, several challenges arise that necessitate a special approach and reasonable assumptions regarding the validation of the deformation model. Uniformly distributed external loads are applied at infinity along the X and Y axes, external, which may be either unequal ($\lambda \neq 1$) or equal ($\lambda = 1$) (λ is the lateral thrust coefficient). The magnitude of these loads is sufficient to generate a plastic deformation zone completely enclosing the excavation boundary. Deformation and failure of the rock mass occur under prescribed deformation conditions from the elastically compressed part of the massif. The assumption of continuity of the medium is preserved in both elastic and plastic zones. Since rock mass movement along the longitudinal axis of the excavation is restricted, the case of plane deformation is considered. To solve the problem, it is necessary to determine stress, strain, and displacement components in both elastic and inelastic regions, as well as the size and shape of the boundary L that separates them.

Key words: underground workings, rock mass, analytical solutions, elastoplastic problems.

STRESZCZENIE: Wokół wyrobisk podziemnych tworzy się ograniczona strefa plastycznie odkształconych skał, wynikająca ze specyficznego połączenia wytrzymałości głównej masywu skalnego, jego struktury oraz głębokości zalegania. Wielkość tej strefy oraz wartość przemieszczeń wzdłuż obrysu wyrobiska determinują jego stateczność. Analityczne rozwiązania zagadnień sprężysto-plastycznych są zazwyczaj ograniczone do uproszczonych modeli ośrodka (ciągłego, izotropowego, jednorodnego) oraz kształtu wyrobiska (kołowego). Matematyczne modelowanie odkształceń sprężysto-plastycznych w strukturalnie niejednorodnym masywie skalnym osłabionym wyrobiskami o złożonym kształcie przeprowadza się wyłącznie z wykorzystaniem metod numerycznych, takich jak metoda elementów skończonych (MES). W tym kontekście pojawia się szereg wyzwań, które wymagają szczególnego podejścia i przyjęcia racjonalnych założeń dotyczących walidacji modelu odkształceń. Na nieskończoności przyłożone są równomiernie rozłożone obciążenia zewnętrzne wzdłuż osi X i Y, które mogą być nierówne ($\lambda \neq 1$) lub równe ($\lambda = 1$) względem siebie, gdzie λ oznacza współczynnik parcia bocznego. Wartość tych obciążeń jest wystarczająca do wygenerowania strefy plastycznej całkowicie otaczającej obrys wyrobiska. Odkształcenia i zniszczenia masywu skalnego następują przy zadanych warunkach odkształceniowych z części masywu poddanego sprężystemu ściskaniu. Założenie ciągłości ośrodka jest zachowane zarówno w strefie sprężystej, jak i plastycznej. Ponieważ ruch masywu skalnego wzdłuż osi podłużnej wyrobiska jest ograniczony, rozpatruje się przypadek płaskiego stanu odkształcenia. W celu rozwiązania zagadnienia konieczne jest wyznaczenie składowych naprężeń, odkształceń i przemieszczeń w strefach sprężystej i niesprężystej, jak również określenie wielkości i kształtu rozdzielającej je granicy L.

Słowa kluczowe: wyrobiska podziemne, masyw skalny, rozwiązania analityczne, zagadnienia sprężysto-plastyczne.

Corresponding author: R.K. Mehtiyev, e-mail: *rafail60mehtiev@mail.ru*

Article contributed to the Editor: 12.03.2024. Approved for publication: 26.05.2025.

Introduction

In general, the collapse of rock masses occurs under the influence of a complex stress state, characterized by a combination of compressive, tensile, and shear stresses. Objective data can be obtained by considering the physical laws governing the scattering process. However, the development of rigorous analytical methods is invariably associated with (sometimes controversial) assumptions and idealization of the rock, due to the complexity and uncertainty of real collapse mechanism. This often results in significant deviations between calculated indicators and actual values. Nevertheless, the analytical approach is superior because it allows consideration of physical laws that operate objectively in nature. A limited area of plastically deformed rocks forms around underground workings, due to specific combination of the strength of the main rock mass, its structure, and burial depth (Chouly and Hild, 2022; Pengpeng and Jun, 2023). The size of this zone and the magnitude of displacement along the excavation boundary determine its stability. Analytical solutions of elastic-plastic problems are typically limited to simplified models of the medium (continuous, isotropic, homogeneous) and excavation shape (circular).

The presence of holes, grooves, recesses and other similar structural or technological formations induces stress concentrations. Plastic deformation zones arise near excavations and openings under sufficiently high external loads. Taking into account plastic zones is especially important in calculating the strength of structures. The complexity of elastoplastic problems lies in the fact that the shape and extent of the plastic region are unknown a priori and must be determined (Hasanov et al., 2020). Existing solutions for plane elastoplastic problems in isotropic bodies with circular holes are associated with full coverage of the circular hole by the plastic deformation region. In such cases, the corresponding mathematical problem of an ideal plastic body is typically reduced to a boundary value problem for a biharmonic equation in a domain with an unknown boundary (Wu et al., 2020; Mirsalimov and Kalandarly, 2021).

Problem Statement

Let us consider the stress-strain state of a homogeneous, isotropic, elastic rock mass near a long, single horizontal excavation of circular cross-section, located at a depth H below of the earth's surface and not influenced by mining operations (Figure 1). The excavation has a radius R_0, and a uniformly distributed load of intensity P_0, equal to the support resistance, is applied to its contour. The rock medium, with a compressive strength R_c, is assumed to be weightless within the zone of influence of the excavation. The greater the excavation depth, the smaller the error resulting from such an idealization; as shown by Mikhlin (1934) and Erzhanov (1959), this error does not exceed 1%.

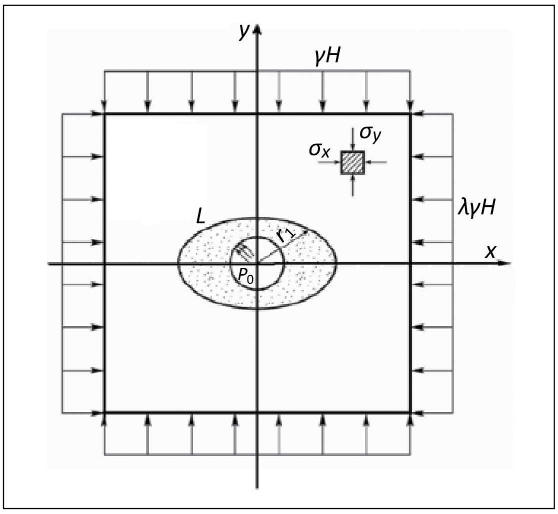

Figure 1. Calculation scheme for solving the problem of equilibrium of a rock mass in the vicinity of a single horizontal excavation (Mirsalimov and Hasanov, 2022)

Rysunek 1. Schemat obliczeniowy do rozwiązania problemu równowagi górotworu w pobliżu pojedynczego wyrobiska poziomego (Mirsalimov i Hasanov, 2022)

The most complex case occurs when the external loads along the horizontal and vertical axes are unequal, i.e., when the lateral thrust coefficient λ is not equal to one ($\lambda \neq 1$).

The calculation scheme shown in Figure 1 is quite general. When tangential stresses exist at infinity (for example, due to neotectonics), the coordinate axes can always be oriented along the main stress directions. Consequently, the stress distribution at infinity will correspond to the assumptions made in the problem.

At an arbitrary point in the rock mass with coordinates X and Y, the stress components satisfy the equilibrium equations:

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0 \quad \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} = 0 \tag{1}$$

the condition of compatibility of deformations:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \times (\sigma_x + \sigma_y) = 0 \tag{2}$$

In the plastic deformation zone, the physical equation is given as:

$$\sigma_\theta + \sigma_r = 2k \left(\frac{A}{r^2} - B \right) \tag{3}$$

where: r, θ – polar radius and angle, respectively,
A, B – constants as determined by equation (58).

Here and below, all length and displacement quantities are related to the excavation radius R_0.

In this case, it is assumed that no tangential stresses exist in the plastic region ($\tau_{r\theta} = 0$), making the stress state axisymmetric.

Let us denote the stress components in the plastic region with the index 1 placed on top, and the stresses in the elastic region without an index.

The boundary conditions are then defined as follows:
On the development contour:

$$\tau_{r\theta}^{1}\Big|_{R=R_0}=0 \qquad \sigma_{r}^{1}\Big|_{R=R_0}=p_0 \tag{4}$$

At infinity:

$$\sigma_x^{\infty}=\lambda\gamma H \qquad \sigma_y^{\infty}=\gamma H \qquad \tau_{xy}^{\infty}=0 \tag{5}$$

At the boundary L between the plastic and elastic regions, the stresses are continuous:

$$\sigma_x^1=\sigma_x \qquad \sigma_y^1=\sigma_y \qquad \tau_{xy}^1=\tau_{xy} \tag{6}$$

A certain strength reduction function $f(r)$ is introduced into the strength condition, which defines how the strength of rocks in uniaxial compression or adhesion changes around the excavation, depending on the relative radius r ($r = R/R_0$, where R_0 is the working radius and R is the current radius).

The principle for selecting an analytical expression for the strength reduction function is essentially uniform. For example, in the "$\sigma - r$" coordinate system, experimental data are approximated by a monotonic curve, the ordinates of which increase from a value near or equal to zero at the excavation boundary to the intact rock strength R_c at the interface between the plastic and elastic regions. To varying degrees, known analytical expressions for the strength reduction function follow this principle. It is evident, however, that if the initial physical model assumes the rock medium to be continuous, then the form of the function $f(r)$ must reflect this assumption. In particular, in both the plastic and elastic regions, the stress function $F(r)$ must be biharmonic. Then it will have a single specific expression.

To determine the type of strength reduction function, we proceed as follows. Let us express the initial relations in polar coordinates. The equations of equilibrium and compatibility of deformations are given as (Timoshenko, 1975; Mirsalimov and Hasanov, 2022):

$$\frac{\partial\sigma_r}{\partial r}+\frac{1}{r}\frac{\partial\tau_{r\theta}}{\partial\theta}+\frac{\sigma_r-\sigma_\theta}{r}=0 \tag{7}$$

$$\frac{1}{r}\frac{\partial\sigma_\theta}{\partial\theta}+\frac{\partial\tau_{r\theta}}{\partial r}+\frac{2\tau_{r\theta}}{r}=0 \tag{8}$$

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial}{\partial\theta^2}\right)\cdot(\sigma_\theta+\sigma_r)=0 \tag{9}$$

where:
r, θ – polar coordinates.

We express the robustness condition in a general form:

$$(\sigma_\theta-\sigma_r)^2+4\tau_{r\theta}^2=4k^2f^2(r) \tag{10}$$

where:
k – a constant dependent on the initial physical parameters in the robustness condition.

Let us define the stress function so that the following relations hold in the plastic region:

$$\sigma_r=\frac{1}{r}\frac{dF}{dr} \qquad \sigma_\theta=\frac{d^2F}{dr^2} \qquad \tau_{r\theta}=0 \tag{11}$$

It is clear that the stress function in this form always satisfies the equilibrium equations.

To determine the analytical expression for the strength reduction function, substitute equations (11) into equations (9) and (10), yielding the system:

$$\frac{1}{r}\frac{dF}{dr}-\frac{d^2F}{dr^2}=\pm 2kf(r) \tag{12}$$

$$\nabla\nabla F=0 \tag{13}$$

where:
∇ – Laplace operator.

Solving equation (12) using the method of variation of constants, we obtain the following expression for the stress function:

$$F(r)=kr^2\int f(r)\cdot r^{-1}dr-k\int rf(r)dr+C_1r^2+C_2 \tag{14}$$

where:
C_1 and C_2 – arbitrary integral constants.

To determine stress components in the plastic region, we use the stress function $F(r)$, related through equations (11), and defined by (12):

$$F(r)=2k\left[r^2\left(C_1+\frac{B}{4}\right)-\frac{B}{2}r^2\ln r-\frac{A}{2}\left(\ln r+\frac{1}{2}\right)\right]+ \\ +C_1r^2+C_2 \tag{15}$$

Applying the second boundary condition on the excavation contour (4), we find the value of the integration constants:

$$C_1=\frac{P_0}{2k}+\frac{A}{4} \qquad C_2=0 \tag{16}$$

Substituting (16), into (15), we obtain:

$$F(r)=2k\left[\frac{r^2}{2}\left(\frac{A}{2}+\frac{B}{2}+\frac{P_0}{k}\right)-\frac{B}{2}r^2\ln r-\frac{A}{2}\left(\ln r+\frac{1}{2}\right)\right] \tag{17}$$

Using expression (17) and formula (15), the stress components in the plastic region become:

$$\begin{aligned} \sigma_r^{(1)}&=\frac{1}{r}\cdot\frac{dF}{dr}=2k\left[\frac{A}{2}\left(1-\frac{1}{r^2}\right)-B\ln r+\frac{P_0}{2k}\right] \\ \sigma_\theta^{(1)}&=\frac{d^2F}{dr^2}=2k\left[\frac{A}{2}\left(1+\frac{1}{r^2}\right)-B(\ln r+1)+\frac{P_0}{2k}\right] \\ \tau_{r\theta}^{(1)}&=0 \end{aligned} \tag{18}$$

Kolosov-Mushkelishvili relations for the elastic zone (Muskhelishvili, 1966; Mirsalimov and Kalantarly, 2021):

$$\sigma_x + \sigma_y = 4Re\Phi(z) \tag{19}$$

$$\sigma_x - \sigma_y + 2i\tau_{xy} = 2\left[\overline{z}\Phi'(z) + \Psi(z)\right] \tag{20}$$

$$2G(u+iv) = (3-4\mu)\int \Phi(z)dz - z\overline{\Phi(z)} - \int \Psi(z)dz \tag{21}$$

where:

$\Phi(z)$ and $\Psi(z)$ – $Z(z = re^{i\psi})$ – analytic functions of the complex plane,

$$G = \frac{E}{2(1+\mu)}$$

where: G – displacement modulus, E – Young's modulus, μ – Poisson's ratio, radial and tangential components of u and v – displacements, respectively; $z = x + iy$.

Let us transform formulas (19) and (20) from Cartesian to polar coordinates, taking into account that $\tau_{r\theta}^{(1)} = 0$:

$$\sigma_x + \sigma_y = \sigma_r + \sigma_\theta \tag{22}$$
$$\sigma_y - \sigma_x + 2i\tau_{xy} = (\sigma_\theta - \sigma_r)e^{-2i\theta}$$

Then, according to statements (6), (18) and (22), the following relations will be true for contour L:

$$4Re\Phi(z) = 2k\left[2A + \frac{P_0}{k} - B(1+2\ln r)\right]$$
$$\overline{z}\Phi'(z) + \Psi(z) = 2k\left(\frac{A}{r^2} - B\right)e^{-2i\theta} \tag{23}$$

If $|z| \to \infty$

$$\Phi(z) = \frac{1}{4}(\sigma_x^\infty + \sigma_y^\infty) + O(z^{-2})$$
$$\Psi(z) = \frac{1}{2}(\sigma_x^\infty - \sigma_y^\infty) + O(z^{-2}) \tag{24}$$

To solve the boundary value problem, we apply the method of Cherepanov, set out in the work of Sokolov (1948). We transition to the parametric plane of the complex variable ζ via the transformation $Z = \omega(\zeta)$. Let us set $\varphi(\xi) = \Phi[\omega(\xi)]$, $\psi(\xi) = \Phi[\omega(\xi)]$. In the adopted notation, using the conjugation condition on L (6), we obtain on the plane ξ the following boundary value problem for determining three unknown functions $\varphi(\xi)$, $\psi(\xi)$, $\omega(\xi)$:

$$\varphi(\xi) + \overline{\varphi(\xi)} = k\left(A - B + \frac{2P_0}{k}\right) - 2kB\ln\sqrt{\omega(\xi)\cdot\overline{\omega(\xi)}} \tag{25}$$

$$\frac{\overline{\omega(\xi)}}{\omega'(\xi)}\varphi'(\xi) + \psi(\xi) = 2k\frac{A - B\left[\omega(\xi)\cdot\overline{\omega(\xi)}\right]}{\left[\omega(\xi)\right]^2} \qquad |\xi| = 1 \tag{26}$$

If $|\xi| \to \infty$

$$\varphi(\xi) = \frac{1}{4}(\sigma_x^\infty + \sigma_y^\infty) + O(\xi^{-2}) \tag{27}$$

$$\psi(\xi) = \frac{1}{2}(\sigma_x^\infty - \sigma_y^\infty) + O(\xi^{-2}) \tag{28}$$

$$\omega(\xi) = O(\xi) \tag{29}$$

In the extended plane ξ, consider the functional equation:

$$\frac{\varphi'(\xi)}{\omega'(\xi)}\overline{\omega}\left(\frac{1}{\xi}\right) + \psi(\xi) = k\frac{A - B\left[\omega(\xi)\overline{\omega}\left(\frac{1}{\xi}\right)\right]}{\left[\omega(\xi)\right]^2} \tag{30}$$

The solution is sought in the following form:

$$\omega(\xi) = C_3\xi + \overline{P}_v\left(\frac{1}{\xi}\right) \tag{31}$$

where $P_v(1/\xi)$ is a polynomial of degree v with undetermined coefficients.

By formally substituting expression (31) into the main equation (30) and expanding all functions into a series in the neighborhood of the point at infinity, we find $v = 1$, hence:

$$\omega(\xi) = C_3\xi + \frac{C_4}{\xi} \tag{32}$$

where:

C_3, C_4 – real unknown constants from the symmetry condition.

To find these constants, consider the functional equation (25) in the extended plane ξ. Let us denote its right side by $f(\xi)$, the exterior of the unit circle with contour L_1 by S^-, and the interior of the unit circle by S^+. Then equation (25) will take the form:

$$\varphi(\xi) + \overline{\varphi(\xi)} = f(\xi) \tag{33}$$

Multiplying each term of expression (33) by the Cauchy kernel and integrating over the contour yields:

$$\frac{1}{2\pi i}\int_{L_1}\frac{\varphi(\xi)}{\xi - z}d\xi + \frac{1}{2\pi i}\int_{L_1}\frac{\overline{\varphi(\xi)}}{\xi - z}d\xi = \frac{1}{2\pi i}\int_{L_1}\frac{f(\xi)}{\xi - z}d\xi \tag{34}$$

where: $z \in S^-$, $\xi \in L_1$.

Since $\phi(\xi)$ is holomorphic outside the contour L_1 and discontinuous on the contour L_1, and the boundary value of the function $z \to \xi$ as $\phi(z)$, we get that expression (34) is equal to the first term:

$$\frac{1}{2\pi i}\int_{L_1}\frac{\varphi(\xi)}{\xi - z}d\xi = 0, \forall Z \in S^- \tag{35}$$

In this case, the function $\varphi(\xi)$ satisfies the conditions of Cauchy's theorem (Lavrentiev and Shabat, 1973) for an infinite domain. Therefore, the second term of expression (34) is equal to:

$$\frac{1}{2\pi i}\int_{L_1}\frac{\overline{\varphi(\xi)}}{\xi - Z}d\xi = -\varphi(Z) + \varphi(\infty) \tag{36}$$

If a function $f(z)$ is holomorphic at S^- and discontinuous at $(S^- + L_1)$, then it is possible that the finite points a_1, a_2, a_3, ..., a_4

of this domain, and also excluding the point $z = \infty$, where $G_1(z)$, $G_2(z)$, ..., $G_n(z)$, $G_\infty(z)$ are poles can, then it can be shown as follows.

If the function $f(z)$ is holomorphic in S^-, continuous in $(S^- + L_1)$ except, perhaps, at the end points a_1, a_2, a_3, ..., a_4 of this region, as well as at $z = \infty$, where it may have a pole with main parts $G_1(z)$, $G_2(z)$, ..., $G_n(z)$, $G_\infty(z)$ then it can be represented as follows:

$$\frac{1}{2\pi i}\int_{L_1}\frac{f(\xi)}{\xi - z}dz = -f(z) + G_1(z) + G_\infty(z) \qquad z \in S^- \tag{37}$$

$$\frac{1}{2\pi i}\int_{L_1}\frac{f(\xi)}{\xi - z}d\xi = \\ = G_1(z) + G_2(z) + \ldots + G_n(z) + G_\infty(z) \qquad z \in S^+ \tag{38}$$

Let's expand the right side of equation (34):

$$\frac{1}{2\pi i}\int_{L_1}\frac{f(\xi)}{\xi - z}d\xi = \\ = \frac{2k}{2\pi i}\int_{L_1}\frac{A - B + \frac{2P_0}{k}}{2(\xi - 2)}d\xi - \frac{2Bk}{4\pi i}\int_{L_1}\frac{\ln\omega(\xi)}{\xi - z}d\xi - \\ - \frac{2Bk}{4\pi i}\int_{L_1}\frac{\ln\overline{\omega(\xi)}}{\xi - z}d\xi \tag{39}$$

The first two expressions in (39) satisfy the conditions (37) and (38), therefore:

$$\frac{1}{2\pi i}\int_{L_1}\frac{A - B + \frac{2P_0}{k}}{2(\xi - z)}d\xi = \\ = \frac{-\left(A - B + \frac{2P_0}{k}\right)}{2} + \frac{\left(A - B + \frac{2P_0}{k}\right)}{2} = 0 \;\text{ if }\; z \to \infty$$

$$\frac{1}{2\pi i}\int_{L_1}\frac{\ln\omega(\xi)}{\xi - z}d\xi = \ln\omega + G_\infty(z) = -\ln\omega + \ln C_3 z = -\ln\frac{\omega}{C_3 z}$$

The third term in expression (39) is equal to zero for the same reason as (35). Thus, we get:

$$-\varphi(z) + \varphi(\infty) = -\frac{1}{2}\ln\frac{\omega(z)}{C_3 z} \qquad \forall z \in S^- \tag{40}$$

From the boundary condition (27) as $z \to \xi$ for the function $\varphi(\xi)$ we find:

$$\varphi(\xi) = 0.25\gamma H(1 + \lambda) - Bk\ln\frac{\omega(\xi)}{C_3\xi} \tag{41}$$

It follows from the equation (30) that:

$$\psi(\xi) = 2k\frac{A - B\left[\omega(\xi)\cdot\overline{\omega(\xi)}\right]}{\left[\omega(\xi)\right]^2} - \frac{\varphi'(\xi)}{\omega'(\xi)}\overline{\omega}\left(\frac{1}{\xi}\right) \tag{42}$$

Considering this:

$$\omega(\xi) = C_3\xi + \frac{C_4}{\xi}, \overline{\omega}\left(\frac{1}{\xi}\right) = \frac{C_3}{\xi} + C_4, \varphi'(\xi) = -\frac{C_4}{\xi(C_3\xi^2 + C_4)}$$

We get:

$$\psi(\xi) = 2k\frac{A - B\left[C_3^2 + C_4^2 + C_3C_4(\xi^2 + \xi^{-2})\right]}{C_3^2\xi^2 + C_4^2\xi^{-2} + 2C_3C_4} + \\ + \frac{C_4(C_4\xi^2 + C_3)}{C_3^2\xi^4 - C_1^2} \tag{43}$$

Thus, the posed problem is solved exactly up to the constants of integration.

Note that as $\xi \to \infty$ $\psi(\xi) = -kB(C_4/C_3)$. On the other hand, from (28) it follows that as $\xi \to \infty$ $\psi(\xi) = 0.5\gamma H(1 - \lambda)$. Equating these two expressions, we get:

$$C_4 = C_3\frac{\gamma H(1 - \lambda)}{2Bk} \tag{44}$$

According to the mean value theorem for harmonic functions:

$$\int_L\frac{\varphi(\xi) + \overline{\varphi(\xi)}}{2\xi}d\xi = 0 \tag{45}$$

Let us conjugate the function of (41) and integrate $R_e f(\xi)$ according to (45). We perform the same procedure for equation (42).

Equating the resulting expressions, we find the constant C_3:

$$C_3 = \exp\left[\frac{A}{2B} - \frac{\gamma H}{4Bk}(1 + \lambda) + \frac{P_0}{2B} - \frac{1}{2}\right] \tag{46}$$

Thus, integral constants are determined. The L boundary between the elastic zone and the collapse zone is an ellipse whose equation is as follows.

$$\frac{X^2}{C_3^2(1 + \beta)^2} + \frac{Y^2}{C_3^2(1 + \beta)^2} = 1 \tag{47}$$

where $\beta = \dfrac{\gamma H(1 - \lambda)}{2Bk}$

For the considered one-dimensional problem, the initial relations in the polar coordinate system are written as follows:

- equilibrium equation:

$$\frac{d\sigma_r}{dr} - \frac{\sigma_\theta - \sigma_r}{r} = 0 \tag{48}$$

- equation of constancy of deformation:

$$\frac{d^2\varepsilon_\theta}{dr^2} + \frac{2}{r}\cdot\frac{d\varepsilon_\theta}{dr} - \frac{1}{r}\cdot\frac{d\varepsilon_r}{dr} = 0 \tag{49}$$

- Hooke's relations:

$$\varepsilon_r = \frac{1}{2G}\left[(1 - \mu)\sigma_r - \mu\sigma_\theta\right] \tag{50}$$

$$\varepsilon_\theta = \frac{1}{2G}\left[(1 - \mu)\sigma_\theta - \mu\sigma_r\right] \tag{51}$$

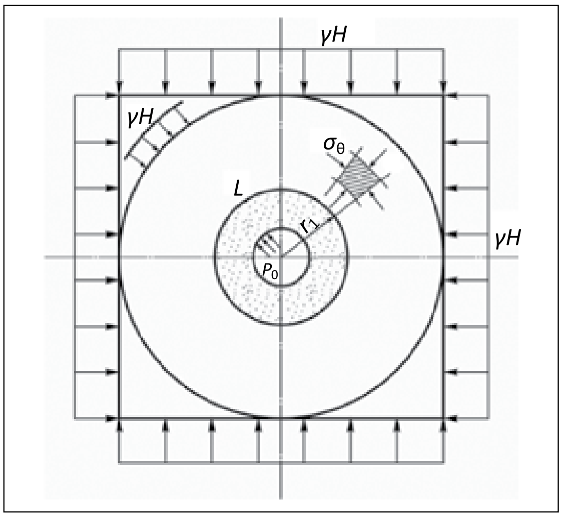

Figure 2. Calculation scheme for solving the problem of equilibrium of a rock mass in the vicinity of a single horizontal excavation (Mirsalimov i Hasanov, 2022)

Rysunek 2. Schemat obliczeniowy do rozwiązania problemu równowagi górotworu w pobliżu pojedynczego wyrobiska poziomego (Mirsalimov and Hasanov, 2022)

- Cauchy relations:

$$\varepsilon_r = \frac{dU}{dr} \qquad \varepsilon_\theta = \frac{U}{r} \tag{52}$$

where:

σ_r, σ_θ and ε_r, ε_θ – radial and tangential components of stresses and strains, respectively,
U – radial displacement,
G – displacement modulus,
μ – Poisson's ratio,
r – polar coordinate.

Here and below, all quantities that have the dimension of length and displacement are still referred to the excavation radius R_0. The boundary conditions and matching conditions are defined as follows:

$$\sigma_r = \sigma_\theta = \gamma\mathrm{H} \quad r \to \infty \tag{53}$$

$$\sigma_r = P_0 \qquad r = 1 \tag{54}$$

$$\sigma_r = \sigma_r^{(1)}, \quad U_r = U_r^{(1)} \quad r = r_L \tag{55}$$

All components of stresses and displacements are denoted without an index in the elastic zone, and with index (1) in the plastic region.

Solving the Euler equation obtained from the expression (53) and satisfying the boundary conditions (49), we obtain the following formulas for the stress components in the elastic zone.

$$\sigma_r = \gamma H - \frac{C}{r^2} \qquad \sigma_\theta = \gamma H + \frac{C}{r^2} \tag{56}$$

where C – unknown integration constant determined from the conditions of conjugation of radial stresses on the contour L (55).

In the zone of inelastic deformations, the following physical equation is valid:

$$\sigma_\theta - \sigma_r = 2k\left(\frac{A}{r^2} - B\right) \tag{57}$$

where:

k – a constant depending on the initial physical conditions included in the strength condition. In our case, it is defined by the expression $\sigma_1 - \sigma_3 = 2k$,
A and B – fixed numbers, determined based on the following expressions.

$$A = \frac{r_L^2}{1 - r_L^2}(1 - k_{ocm}) \qquad B = \frac{r_L^2 - k_{ocm}}{1 - r_L^2} \tag{58}$$

where:

r_L – dimensionless radius of the inelastic deformation zone,
k_{ocm} – residual strength factor.

Solving this equation together with the equilibrium equation (48), and considering the boundary conditions (54), we obtain expressions for the stress components in the plastic region:

$$\sigma_r^{(1)} = -2k\left[0.5A(r^{-2} - 1) + B\ln r\right] + P_0 \tag{59}$$

$$\sigma_\theta^{(1)} = -2k\left[0.5A(r^{-2} + 1) + B\ln r\right] + P_0 \tag{60}$$

Considering the equality of the radial stresses determined by formulas (56) and (59) for $r = r_L$, the value of the unknown integral constant is found to be $C = kr_L^2$.

Thus, the stress components in the elastic and plastic zones are determined. Then, using expressions (55), (59), we obtain a transcendental expression for determining the radius of the zone of inelastic deformations:

$$0.5A(r_L^{-2} - 1) + B\ln r_L = \frac{\gamma H - P_0}{2k} - \frac{1}{2} \tag{61}$$

From the expression (61) it follows that, firstly, the protection of support P_0 has a negligible effect on the size of the inelastic deformation region, since its value at deep mine levels γH is much smaller than the gravity pressure. Accordingly, we can define $P_0 = 0$ in formula (61) without loss of accuracy. Secondly, for the most coal-bearing rocks, the value of ψ included in the following expression is approximately 0.1.

From (61) it follows, firstly, that the resistance of the support P_0 has an extremely small effect on the size of the region of inelastic deformations, since its value at the deep mine levels is incomparably smaller than the gravitational pressure γH. In this regard, we can set $P_0 = 0$ in formula (61) without compromising accuracy. Secondly, for the vast majority of coal-bearing rocks, the value ψ in the expression $(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2 - R_C^2\psi - (1-\psi)R_C(\sigma_x + \sigma_y) = 0$ is approximately 0.1. If we set it equal to zero, the error from such an idealization will not exceed 5%.

If $\psi = 0$, the error from such an idealization will not exceed 5%.

Based on the analysis of the dependency $r_L = f(R_c k_c/\gamma H)$ for different values of the residual strength coefficient k_{ocm}, presented in the study of Shashenko et al. (2001), we assume $k_{ocm} = 0$. Then, based on expression (54), the following formula derived from equation (60) is used to determine the radius of the inelastic deformation zone:

$$\frac{r_L^2 \ln r_L}{r_L^2 - 1} = \sqrt{\frac{\gamma H}{R_c k_c}} \tag{62}$$

Using the Cauchy relations (52) and the smoothing function $f'(r) = 1 + B - Ar^{-2}$, taking assuming that $\varepsilon_r + \varepsilon_\theta = \varepsilon_v$, the following inhomogeneous differential equation is obtained:

$$\frac{dU}{dr} + \frac{U}{r} = \varepsilon_v^* \left(1 + B + \frac{A}{r^2}\right) \tag{63}$$

where:

ε_v^* – limiting volumetric strain under uniaxial compression conditions.

The solution of the corresponding homogeneous equation has the following form:

$$U = C \cdot r^{-1} \tag{64}$$

Taking into account the equality of radial displacements in the contour L and changing the constant yields the following expression for determining displacements in the plastic region:

$$U = \frac{\varepsilon_v^*}{2r}\left[(B+1)\cdot(r^2 - r_L^2) - 2A\ln\frac{r}{r_L}\right] \tag{65}$$

Considering $k_{ocm} = 0$ in expressions (62) and (58), we obtain the following expression for determining displacements along the contour of the excavation:

$$U_0 = \varepsilon_v^* \left(0.5 - \sqrt{\frac{\gamma H}{R_c k_c}}\right) \tag{66}$$

The main dependencies for determining the elastic-plastic state parameters of the rock mass in the vicinity of a single excavation, obtained above in (62) and (66), make it possible

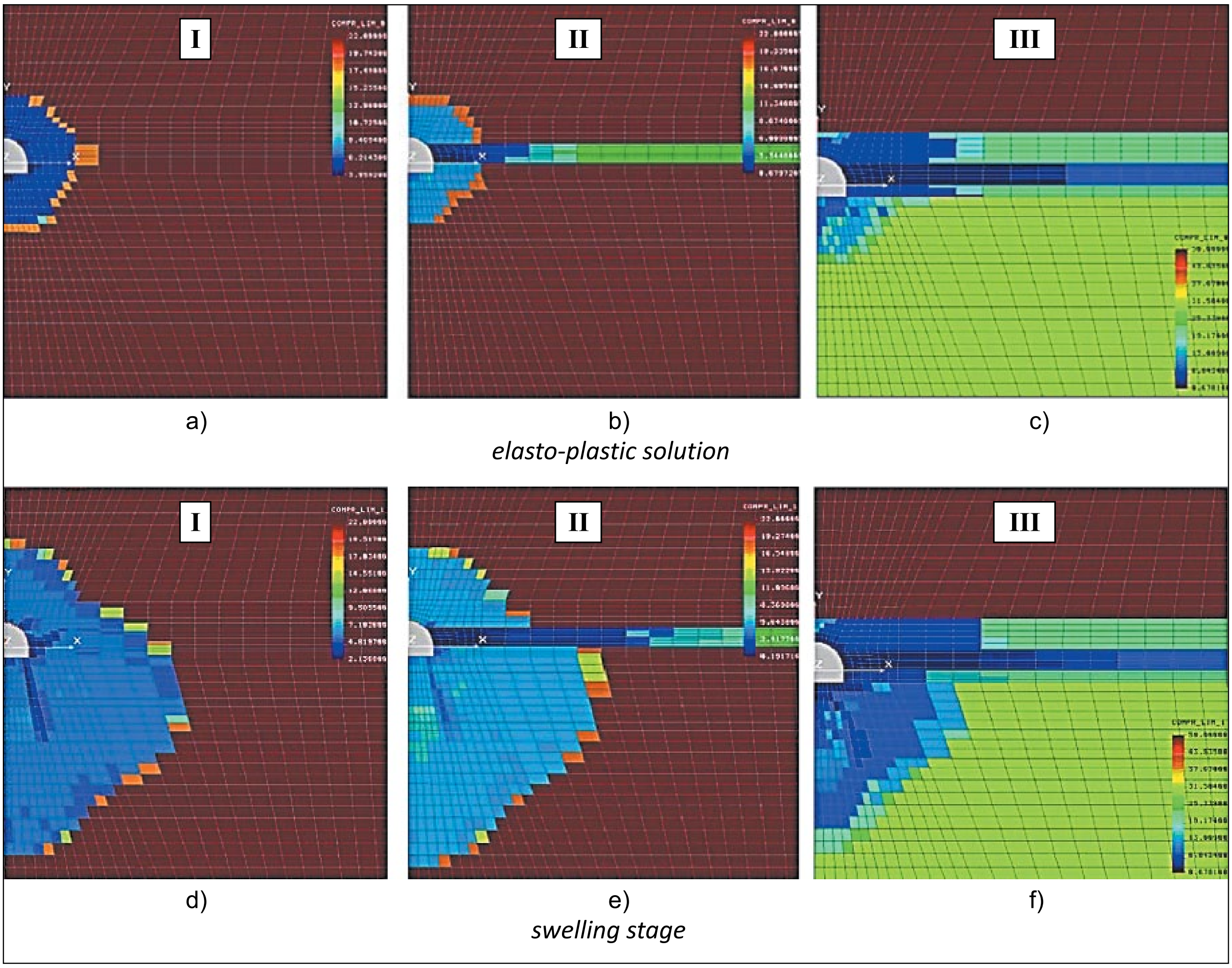

Figure 3. Configurations of zones of inelastic deformation at different degrees of heterogeneity of the rock mass: I – homogeneous rock mass; II – massif including a coal seam; III – layered massif

Rysunek 3. Konfiguracje stref deformacji niesprężystej przy różnym stopniu niejednorodności górotworu: I – jednorodny górotwór; II – masyw, w tym pokład węgla; III – masyw warstwowy to determine the point values of probabilistic quantities: the radius of the inelastic deformation region r_L, and the radial displacement on the excavation contour U_0.

Let us determine the radius of the inelastic deformation zone and the magnitude of displacements along the excavation contour under the following average initial conditions:

- excavation depth, $H = 350$ m;
- uniaxial compressive strength, $\sigma_c = 25$ MPa;
- bulk density, $\gamma = 2.50 \cdot 10^{-3}$ MH/m^3;
- excavation radius, $R_c = 2.0$ m;
- coefficient of structural-mechanical weakening, $k_c = 0.33$;
- limiting volumetric deformation under uniaxial compression, $\varepsilon_v^* = -0.1$.

According to expressions (62) and (66), under these conditions, we obtain:

$$r_L = 2.3, \text{ and } U_0 = 0.38 \text{ m}$$

Analysis of Figure 3 reveals a significant influence of the rock properties and the structure of the massif surrounding the excavation on the formation of the inelastic deformation zone, and, consequently, on the stability of the excavation. This effect is especially pronounced under conditions of soil extrusion (Figure 3d, e, f), where a reduction in the size of the inelastic deformation zone is observed in the case of a layered massif (Shashenko et al., 2008).

Conclusion

The final expressions of the solution are highly complex, which complicates their research and practical applications. It is concluded that within the upper layer of the lithosphere, where mining is typically conducted, in horizontally bedded sedimentary rocks and under a wide range of mining and geological conditions, the stress distribution in the undisturbed rock mass may be assumed to be hydrostatic, i.e. $\lambda = 1$. In this case, the solution to the formulated problem is considerably simplified, as the elliptical contour degenerates into a circle. The corresponding calculation scheme used to solve the problem is presented in Figure 2.

References

Chouly F., Hild P., 2022. On a finite element approximation for the elastoplastic torsion problem. *Applied Mathematics Letters*, 132: 108191.

Erzhanov Zh.S., 1959. Study of creep in Donbass rocks. *TsITuglya*, 1: 36.

Hasanov I., Abbasov I., Gurbanov N., 2020. Stress-Deformed State of a Packing Ring with Eccentric Holes. *Proceedings of the Latvian Academy of Sciences. Section B. Natural, Exact, and Applied Sciences*, 74(4): 287–292. DOI: 10.2478/prolas-2020-0044.

Lavrentiev M.A., Shabat E.V., 1973. Methods in the theory of functions of a complex variable. *Nauka*, 736.

Mikhlin S.G., 1934. Stress distribution in a half-plane with an elliptical cutout. *Tr. seismic Institute of the USSR Academy of Sciences*, 29: 212–220.

Mirsalimov V.M., Hasanov F.F., 2022. Solution of elastoplastic problem for a cracked massif weakened by a circular hole. *News of Tula State University Geosciences*, 1: 332–345.

Mirsalimov V.M., Kalantarly N.M., 2021. Solution of an elastoplastic problem for a massif weakened by a circular mining under the action of tectonic and gravitational forces. *Proceddings of Tula State University. Sciences of Earth*, 1: 207–216.

Muskhelishvili N.I., 1966. Some basic problems of the mathematical theory of elasticity. *Science*, 705.

Pengpeng Sh., Jun X., 2023. Exact solution of magneto-elastoplastic problem of functionally graded cylinder subjected to internal pressure. *Applied Mathematical Modelling*, 123(7): 835–855. DOI: 10.1016/j.apm.2023.08.014.

Shashenko A.N., Sdvizhkova E.A., Gapeev S.N., 2008. Deformability and strength of rocks. *Dnepropetrovsk NSU*, 225.

Shashenko A.N., Tulub S.B., Sdvizhkova E.A., 2001. Some problems of statistical geomechanics. *Pulsari*, 243.

Sokolov A.P., 1948. On the elastic-plastic state of the plate. *DAN USSR*, 1: 128–139.

Timoshenko S.P., 1975. Goodyear J. Theory of Elasticity. *Science*, 576.

Wu Q., Liu F.B., Cheng Y.M., 2020. The interpolating element-free Galerkin method for three-dimensional elastoplasticity problems. *Engineering Analysis with Boundary Elements*, 115: 156–167. DOI: 10.1016/j.enganabound.2020.03.009.

Rafail Karim MEHTIYEV, Ph.D.
Assistant Professor at the Department of General and Applied Mathematics
Azerbaijan State Oil and Industry University
16/21 Azadliq Ave, AZ1010 Baku, Azerbaijan
E-mail: *rafail60mehtiyev@mail.ru*

Yusif Alovsat TANRIVERDIYEV, M.Sc.
Teacher at the Department of Material Science and Processing Technology
Azerbaijan State Oil and Industry University
16/21 Azadliq Ave, AZ1010 Baku, Azerbaijan
E-mail: *yusif.tanriverdiyev@asoiu.edu.az*