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Derivation of the modified Blake-Kozeny equation for Herschel-Bulkley fluids

Wyprowadzenie zmodyfikowanego równania Blake'a-Kozeny'ego dla cieczy typu Herschela-Bulkley’a

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ABSTRACT: Some heavy oil fields that have significant reserves in the world are difficult to develop due to the non-Newtonian nature of those oils. The importance of this development is increasing due to the depletion of conventional oil reservoirs. These oils do not start flowing immediately at a pressure gradient greater than zero, but only after exceeding the yield point. Instead, the pressure gradient must exceed a certain value which depends on the yield stress of the oils. This work aimed to derive the modified Blake-Kozeny equation for non-Newtonian fluids, the flow of which can be described by the Herschel-Bulkley model. The modified Blake-Kozeny equation is an alternative to Darcy’s law, which can be used to describe the flow of non-Newtonian oils in porous media. The equation derived is strongly non-linear and complicated to use. The results obtained using this equation have been compared with those obtained using a simplified semi-empirical equation widely used in the literature, and the results were very close. Therefore, this work theoretically justifies the simplified equation by demonstrating its accuracy as an approximation of the fully analytical solution over practical parameter ranges. An explicit formula for the flow condition of yield stress fluids in porous media has been derived, allowing the determination of the empirical coefficient by using the simplified equation which is the critical pressure gradient.

Keywords: non-Newtonian fluids; Herschel-Bulkley model; Blake-Kozeny equation.

STRESZCZENIE: Niektóre złoża ropy ciężkiej, dysponujące znacznymi zasobami w skali światowej, są trudne do eksploatacji ze względu na nienewtonowski charakter takiej ropy. Znaczenie ich eksploatacji rośnie ze względu na wyczerpywanie się konwencjonalnych złóż ropy. Ropa tego rodzaju nie zaczyna płynąć natychmiast przy gradiencie ciśnienia większym od zera, ale dopiero po przekroczeniu granicy płynięcia. Oznacza to, że gradient ciśnienia musi przekroczyć określoną wartość, która zależy od naprężenia granicznego ropy. Celem niniejszej pracy było wyprowadzenie zmodyfikowanego równania Blake’a-Kozeny’ego dla cieczy nienewtonowskich, których przepływ można opisać modelem Herschela-Bulkley’a. Zmodyfikowane równanie Blake’a-Kozeny’ego jest alternatywą dla prawa Darcy’ego, które można wykorzystać do opisu przepływu ropy nienewtonowskiej w ośrodkach porowatych. Otrzymane równanie ma silnie nieliniowy charakter i jest skomplikowane w użyciu. Wyniki uzyskane przy użyciu tego równania porównano z wynikami uzyskanymi przy użyciu uproszczonego równania półempirycznego, szeroko stosowanego w literaturze. Stwierdzono bardzo dobrą zgodność obu podejść. W związku z tym niniejsza praca stanowi teoretyczne uzasadnienie stosowania uproszczonego równania, wykazując jego wysoką dokładność jako przybliżenia pełnego rozwiązania analitycznego w praktycznie istotnych zakresach parametrów. Wyprowadzono również wyraźną zależność opisującą warunek przepływu cieczy o granicy płynięcia w ośrodkach porowatych, co umożliwia wyznaczenie współczynnika empirycznego przy użyciu uproszczonego równania, którym jest krytyczny gradient ciśnienia.

Słowa kluczowe: ciecze nienewtonowskie, model Herschela-Bulkley’a, równanie Blake’a-Kozeny’ego.

Introduction

Due to the recent growth in demand for oil and gas, production of these hydrocarbons has also increased, leading to a significant decrease in conventional oil reserves. As a result, most oil reserves today are non-conventional (Yatimi et al., 2024). Around 70% of the world’s oil reserves are non-conventional (Muther et al., 2021). Therefore, studying their physical and chemical properties is of great practical interest (Song et al., 2015). Some experts predict that most future energy demand will be met by the extraction of non-conventional oil reserves (Zyrin and Ilinova, 2016). The term “non-conventional hydrocarbon reserves” means tight oils, shale oils, bitumens, oil sands, and highly viscous heavy oils that have non-Newtonian behavior (Cander, 2012).

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Most non-Newtonian fluids have high viscosity, which decreases their flow rate, so the production of non-Newtonian oils may become economically unviable (Rana et al., 2017). To make the production of non-Newtonian oils profitable, their viscosity needs to be reduced, which can be achieved by increasing the reservoir temperature (Qin et al., 2018). Enhanced oil recovery methods that focus on decreasing oil viscosity by increasing its temperature are designated as thermal methods. The most commonly used thermal method is the injection of hot steam into the reservoir (Ahmadi and Chen, 2020). It includes the highly popular Cyclic Steam Stimulation (CSS) and Steam-assisted Gravity Drainage (SAGD) (Law, 2016).

Many authors have studied non-Newtonian fluids. Kelbaliyev et al. (2022a) analyzed the problems of non-Newtonian oil rheology, which include physical phenomena of formation and destruction of coagulation structures and aggregates, which have a significant impact on the flow of these fluids. Uscilowska (2008) has used the Brinkman equation, which is a modification of Darcy's law, to consider the non-Newtonian properties of oils. Based on this equation, a simulation of non-Newtonian fluid flow through porous media was performed, and the pressure and velocity distributions in these media were presented. Novruzova (2019) conducted an in-depth study of numerous methods enhanced oil recovery methods used in reservoirs with non-Newtonian oil. Experimental studies related to the determination of the initial pressure gradient that corresponds to the occurrence of yield stress in non-Newtonian fluids have been conducted by Gafarov (2005).

The non-Newtonian properties of crude oils during reservoir development are important for numerous reasons. First, since these fluids do not follow Darcy's law, applying the traditional law during reservoir modeling would yield inaccurate predictions. Many non-Newtonian fluids are characterized by the presence of yield stress, which is the minimal shear stress that must be applied to the fluid to cause it to flow. This means the pressure gradient must also exceed a certain value. During development, this leads to formation of dead zones within the reservoir. Such a zone is far enough from the production well that the pressure gradient is too small, so that no flow can occur. Ignoring the non-Newtonian properties during reservoir simulation and modeling would overestimate the final oil recovery factor (Nikitin, et al., 2020).

Most low-molecular-weight fluids such as water and air obey Newton's law, which is defined in equation (1). These are called Newtonian fluids (Chhabra, 2010):

$$\tau = \mu\dot{\gamma} \tag{1}$$

The main characteristic of Newtonian fluids is that the relationship between shear stress τ and shear rate $\dot{\gamma}$ is linear. The constant of proportionality (μ) is termed "dynamic viscosity". However, not all fluids obey Newton's law; these are called non-Newtonian fluids. Figure 1 illustrates the relationship between shear stress and shear rate for various types of non-Newtonian fluids.

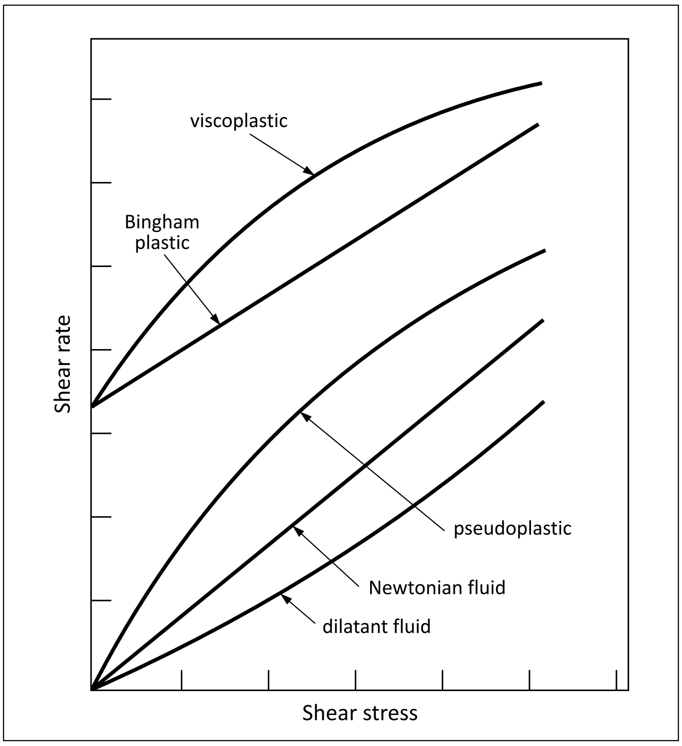

Figure 1. Different types of non-Newtonian fluids
Rysunek 1. Różne rodzaje cieczy nienewtonowskich

All fluids in Figure 1 are considered time-independent. Importantly, some sets of non-Newtonian fluids are time-dependent, for example, the Maxwell model (Sochi, 2010). However, these are outside the scope of this work. Equations (2) (Bingham Plastic model), (3) (Power Law model), (4) (Herschel-Bulkley model) (Alderman, 1997) show the most commonly used time-independent non-Newtonian fluid models:

$$\tau = \tau_0 + \mu_B\,\dot{\gamma} \tag{2}$$

$$\tau = K\dot{\gamma}^n \tag{3}$$

$$\tau = \tau_0 + K\dot{\gamma}^n \tag{4}$$

Not many publications study Bingham fluids and Herschel-Bulkley fluids due to the presence of yield stress, which brings mathematical complexities, for example, when considering unsteady-state flow. The behavior of Bingham fluids in porous media has been studied by Wu (1990). Power law fluids have been deeply studied by: Chhabra et al. (2001), Lopez et al. (2003), Liu and Masliyah (1998). These works analyzed a modified Darcy's law for Power Law fluids and used it to derive and solve the modified diffusivity equation for Power Law fluids under unsteady-state pressure conditions. However, only a few publications have studied the flow of Herschel-Bulkley and Bingham fluids in porous media. This work fills this gap and proposes an analytically derived equation that describes the flow of such fluids in porous media.

The focus of this article is the use of the Herschel-Bulkley equation, because it is the most general equation that includes both the PL (Power Law) and BP (Bingham Plastic) laws. Before 2006, the Herschel-Bulkley model was not widely used, as it was more complex than Power Law and Bingham Plastic models, with no evidence that it was more accurate. However, in 2006 the American Petroleum Institute (API) declared that the Herschel-Bulkley model was the most accurate model for drilling fluids and recommended it for use in the petroleum industry. This has led to an increase in applications of the HB (Herschel-Bulkley) model (Ashena et al., 2023).

Mitchell and Miska (2011) have shown the velocity profile for an HB model in a pipe however, they did not provide the derivation of this profile. The aim of the work by Ashena et al. (2023) was to provide a complete mathematical derivation of the HB fluid velocity profile in a pipe. The main complexity of the solutions provided by Ashena et al. (2023) and Mitchell and Miska (2011) lay in solving the equation for the pressure drop. Gjerstad and Time (2015) and Merlo et al. (1995) presented two different approximation equations for the pressure drop of HB fluid. Simplified flow equations for the Herschel-Bulkley fluid were presented by Gjerstad and Time (2015).

An experimental study to find the relationship between shear stress and shear rate for non-Newtonian fluids was conducted by Chevalier et al. (2013). The porous media was formed by glass bead packings with a diameter D ranging from 0.26 to 2 mm. The authors used two fluids in the experiments: Carbopol (U980) solution in water and water-in-oil emulsion. Both were fluids with a yield-point. The authors concluded that for both fluids, Herschel-Bulkley was the most suitable model. For the Carbopol solution the HB coefficients were $8 < \tau_0 < 89$ Pa; $3.6 < K < 49$ Pa $\cdot$ s^n; $n = 0.36$. For the water-in-oil emulsion the coefficients were: $54 < \tau_0 < 71$ Pa; $12.7 < K < 13.4$ Pa $\cdot$ s^n; $n = 0.36$. These ranges are provided because various concentrations were analyzed.

Ouyang (2013) has analyzed the flow of non-Newtonian Herschel-Bulkley fluid in a packed bed (cubic packing) of spherical proppant particles. Velocity profiles (both along and perpendicular to the flow direction) in the porous media were obtained using numerical simulation. Ouyang (2013) has also provided relationship between pressure gradient and velocity for various power law indexes. The relationship between shear rate and shear stress has also been studied experimentally by Danov (2001) and Argillier et al. (2002) for oils that contain asphaltenes and, thus exhibit non-Newtonian behavior. Kelbaliyev et al. (2022b) determined that for such oils the empirical relationship between shear stress and shear rate is:

$$\tau = \tau_0 + 45.76\dot{\gamma}^{1.75} \tag{5}$$

This also confirms that the most suitable mathematical model for non-Newtonian asphaltic oils is the Herschel-Bulkley model.

Various mathematical models have been proposed relating the apparent velocity (also known as Darcy velocity) in the porous media to the pressure drop. The most widely used relationship is Darcy's law:

$$v = -\frac{k}{\mu}\frac{dP}{dx} \tag{6}$$

Another common equation that relates apparent velocity to pressure gradient is known as Blake-Kozeny equation (Bird, et al., 1960). The derivation of equation (7) below is given in Appendix A:

$$v = -\frac{D_p^2\phi^3}{150\mu(1-\phi)^2}\frac{dP}{dx} \tag{7}$$

The Blake-Kozeny equation is one of the most successful equations that describe the flow of Newtonian fluid in porous media (Christopher and Middleman, 1965). Comparing the Blake-Kozeny equation with Darcy's law leads to the conclusion that:

$$k = \frac{D_p^2\phi^3}{150(1-\phi)^2} \tag{8}$$

Balhoff (2005) demonstrated that the general relationship between permeability and particle diameter D_p is:

$$k = \frac{D_p^2\phi^3}{72C(1-\phi)^2} \tag{9}$$

where C is the tortuosity constant, which is often taken as 25/12 (Blake-Kozeny equation), however, this may vary depending on the geometry of porous media.

Another equation that relates apparent velocity to the pressure gradient is the Ergun equation, however, it is less common than the Darcy and Blake-Kozeny equations (Stevenson, 2003):

$$\frac{dP}{dx} = \frac{150\mu v(1-\phi)^2}{D_p^2\phi^3} + \frac{1.75\rho v^2(1-\phi)}{D_p\phi^3} \tag{10}$$

In equation (10), the first term corresponds to the Blake-Kozeny equation (the viscous term), while the second is the inertial term. At low velocities, the first term is dominant, and the second can be ignored. However, at higher velocities, the second term becomes more important and cannot be ignored. In most oil and gas applications, the reservoir velocities do not usually reach such a high value that the inertial term should be considered; thus – the second term can be ignored.

Pascal (1984) proposed the following modified Darcy's law for Herschel-Bulkley fluids:

$$v = \left[\frac{k}{\mu_{ef}}\left(-\frac{\partial P}{\partial x} - \alpha_0\right)\right]^{\frac{1}{n}} \tag{11}$$

where k/μ_{ef} is defined as:

$$\frac{k}{\mu_{ef}} = \frac{1}{2K}\left(\frac{n\phi}{1+3n}\right)^{n}\left(\frac{8k}{\phi}\right)^{\frac{n+1}{2}} \tag{12}$$

and α_0 is defined as:

$$\alpha_0 = \frac{\beta\tau_0}{\sqrt{k}} \tag{13}$$

where β is a constant that should be determined experimentally.

A similar equation has been proposed by Mirzajanzade for Bingham fluids flowing in porous media (Khusainova and Kashapov, 2019):

$$v = \frac{k}{\mu_B}\left(-\frac{\partial P}{\partial x} - \alpha_0\right) \tag{14}$$

In equations (11) and (14), the use of the expression $-\partial P/\partial x - \alpha_0$ is not analytically derived. Instead, the formulas have been proposed by the corresponding authors (without derivation) to take into account yield stress behavior, which the experiments have shown to be accurate.

Christopher and Middleman (1965) have derived a modified Blake-Kozeny equation for the Power Law fluids. Their strategy was as follows:

1. They assumed that the permeability equation (8) defined for the Blake-Kozeny equation derived for Newtonian fluids, should also apply to non-Newtonian fluids, because the relationship between permeability, porosity and particle diameter should be independent of fluid type.
2. They considered that the Blake-Kozeny equation was derived from the Hagen–Poiseuille equation, which is true for Newtonian fluids and gives the relationship between the average velocity and pressure gradient in a pipe.
3. They derived a relationship between average velocity and pressure gradient in a pipe for Power Law fluids (analogous to the Hagen–Poiseuille equation, but for Power Law fluids).
4. They transformed the actual average velocity to Darcy velocity using the steps given in Appendix A (steps for Newtonian fluids).
5. Finally, they obtained the modified Blake-Kozeny equation for Power Law fluids given below:

$$v = \left(\frac{k}{\mu_{ef}}\frac{dP}{dx}\right)^{\frac{1}{n}} \tag{15}$$

where μ_{ef} is defined as:

$$\mu_{ef} = \frac{K}{12}\left(9+\frac{3}{n}\right)^{n}\left(150k\phi\right)^{\frac{1-n}{2}} \tag{16}$$

In this work, the same assumptions were made and the same steps were taken as Christopher and Middleman (1965) with regard to PL fluids to derive a modified Blake-Kozeny equation for Herschel-Bulkley fluids and analyze the properties of the obtained equation.

Derivation of the modified Blake-Kozeny equation for HB fluid

Equation (17) was obtained by Ashena, et al. (2023) and earlier by Mitchell and Miska (2011) and it shows the flow rate of Herschel-Bulkley fluids flowing in a pipe with radius R.

$$Q = \frac{\pi n}{K^{\frac{1}{n}}}\frac{\left(-\frac{R}{2}\frac{dP}{dz}-\tau_0\right)^{1+\frac{1}{n}}}{\left(-\frac{1}{2}\frac{dP}{dz}\right)^{3}} \cdot \left(\frac{\left(-\frac{R}{2}\frac{dP}{dz}-\tau_0\right)^{2}}{3n+1} + \frac{2\tau_0\left(-\frac{R}{2}\frac{dP}{dz}-\tau_0\right)}{2n+1} + \frac{\tau_0^2}{n+1}\right) \tag{17}$$

Based on equation (17), the average velocity in a pipe is equal to:

$$v = \frac{n}{K^{\frac{1}{n}}R^2}\frac{\left(-\frac{R}{2}\frac{dP}{dz}-\tau_0\right)^{1+\frac{1}{n}}}{\left(-\frac{1}{2}\frac{dP}{dz}\right)^{3}} \cdot \left(\frac{\left(-\frac{R}{2}\frac{dP}{dz}-\tau_0\right)^{2}}{3n+1} + \frac{2\tau_0\left(-\frac{R}{2}\frac{dP}{dz}-\tau_0\right)}{2n+1} + \frac{\tau_0^2}{n+1}\right) \tag{18}$$

We now convert this velocity to Darcy velocity by doing the following:

- Multiply equation (18) by porosity ϕ to take into account that the Darcy velocity is lower than the actual velocity in the pores;
- Multiply the term dP/dz in the equation (18) by $1/c$. This is done to account for the tortuosity of the porous media. Tortuosity increases the actual velocity compared to Darcy velocity;
- As shown in Appendix A, the relationship between hydraulic radius and pipe radius is $r_H = R/2$. The average hydraulic radius of the porous media can be defined as $r_H = (D_p\phi)/(6(1-\phi))$. Therefore, R is approximately equal to $R = (Dp\,\phi)/(3(1-\phi))$. Comparing this with equation (9) demonstrates that $R = \sqrt{(8kC/\phi)}$.

Inserting this definition into Equation (18) transforms it into:

$$v = \frac{\phi n}{K^{\frac{1}{n}}\left(\frac{8kC}{\phi}\right)^{2}}\frac{\left(-\frac{\sqrt{\frac{8kC}{\phi}}}{2C}\frac{dP}{dz}-\tau_0\right)^{1+\frac{1}{n}}}{\left(-\frac{1}{2C}\frac{dP}{dz}\right)^{3}} \cdot$$

$$\cdot\left(\frac{\left(-\frac{\sqrt{\frac{8kC}{\phi}}}{2C}\frac{dP}{dz}-\tau_0\right)^2}{3n+1}+\frac{2\tau_0\left(-\frac{\sqrt{\frac{8kC}{\phi}}}{2C}\frac{dP}{dz}-\tau_0\right)}{2n+1}+\frac{\tau_0^2}{n+1}\right) \qquad (19)$$

Simplifying equation (19) yields:

$$v=\frac{\phi^2 n}{8K^{\frac{1}{n}}kC}\frac{\left(-\sqrt{\frac{2k}{\phi C}}\frac{dP}{dz}-\tau_0\right)^{1+\frac{1}{n}}}{\left(-\frac{1}{2C}\frac{dP}{dz}\right)^3}\cdot$$

$$\cdot\left(\frac{\left(-\sqrt{\frac{2k}{\phi C}}\frac{dP}{dz}-\tau_0\right)^2}{3n+1}+\frac{2\tau_0\left(-\sqrt{\frac{2k}{\phi C}}\frac{dP}{dz}-\tau_0\right)}{2n+1}+\frac{\tau_0^2}{n+1}\right) \qquad (20)$$

Equation (20) is the modified Blake-Kozeny equation for Herschel-Bulkley fluids which can be used to describe the flow of such fluids in a porous media.

Let us now view special cases of this equation. First, by assuming that $\tau_0 = 0$, $n \neq 1$:

$$v=\frac{\phi^2C^2n}{(3n+1)kK^{\frac{1}{n}}}\left(\frac{2k}{\phi C}\right)^{\frac{3n+1}{2n}}\left(-\frac{dP}{dz}\right)^{\frac{1}{n}} \qquad (21)$$

Assuming that $C = 25/12$, as Christopher and Middleman (1965) did based on conducted experiments, then equation (21) will yield the same results as equation (15). In other words, equation (20) as derived for HB fluids, transforms into the equation for PL fluids when setting $\tau_0 = 0$. Setting $n = 1$ (Newtonian fluid case) transforms equation (21) into classic Darcy's law:

$$v=-\frac{k}{K}\frac{dP}{dz} \qquad (22)$$

Setting $n = 1$, $\tau_0 \neq 0$ (Bingham plastic fluid case) in equation (20) yields the modified Blake-Kozeny equation for Bingham plastics:

$$v=\frac{\phi^2C^2}{kK}\frac{1}{\left(-\frac{dP}{dz}\right)^3}\cdot$$

$$\cdot\left(\frac{\left(-\sqrt{\frac{2k}{\phi C}}\frac{dP}{dz}-\tau_0\right)^4}{4}+\frac{2\tau_0\left(-\sqrt{\frac{2k}{\phi C}}\frac{dP}{dz}-\tau_0\right)^3}{3}+\frac{\tau_0^2\left(-\sqrt{\frac{2k}{\phi C}}\frac{dP}{dz}-\tau_0\right)^2}{2}\right) \qquad (23)$$

Further simplification of equation (23) leads to the equation below:

$$v=\frac{k}{K}\left(-\frac{dP}{dz}-\frac{2}{3}\sqrt{\frac{2\phi C}{k}}\tau_0+\frac{\phi^2C^2}{12k^2}\tau_0^4\frac{1}{\left(-\frac{dP}{dz}\right)^3}\right) \qquad (24)$$

In equation (24) the first term corresponds to the term from Darcy's classical law, while the second and third terms are related to the presence of yield stress. The third term makes this equation more complicated as it makes it non-linear. Analysis of equation (20) demonstrates that for a non-Newtonian fluid with a yield point to flow in porous media, the pressure gradient must exceed the following value:

$$-\frac{dP}{dz}>\sqrt{\frac{\phi C}{2k}}\tau_0\approx\sqrt{\frac{\phi}{k}}\tau_0 \qquad (25)$$

Now, the initial pressure gradient α_0 used in equation (11) becomes:

$$\alpha_0=\sqrt{\frac{\phi C}{2k}}\tau_0\approx\sqrt{\frac{\phi}{k}}\tau_0 \qquad (26)$$

Therefore, by comparing equation (26) with equation (13), parameter β which has been labeled as "the one to be determined experimentally" in previous literature, can be expressed as:

$$\beta=\sqrt{\frac{\phi C}{2}}\approx\sqrt{\phi} \qquad (27)$$

Results and Discussion

We performed analysis and comparison of the relationship between velocity and the pressure gradient based on a simplified semi-empirical equation (11) with a relationship based on a much more complex, fully analytical equation (20) derived in this work, using the values of the parameters in Table 1.

Table 1. Parameters used for visualization and analysis of equations (11) and (20)

Tabela 1. Parametry wykorzystane do wizualizacji i analizy równań (11) i (20)

Parameter [unit]	Parameter value
τ_0, [Pa]	0.3
K, [Pa · s^n]	5×10^{-3}
C	25/12
ϕ	0.1

Figure 2 shows the relationship between velocity and the pressure gradient for permeability $k = 10^{-13}$ m^2 built for various values of n for equation (20) derived in this work (colored legend, solid lines) and the relationship between velocity and the pressure gradient for various values of n for the simplified equation (11) (illustrated by a black dotted line). Firstly, this

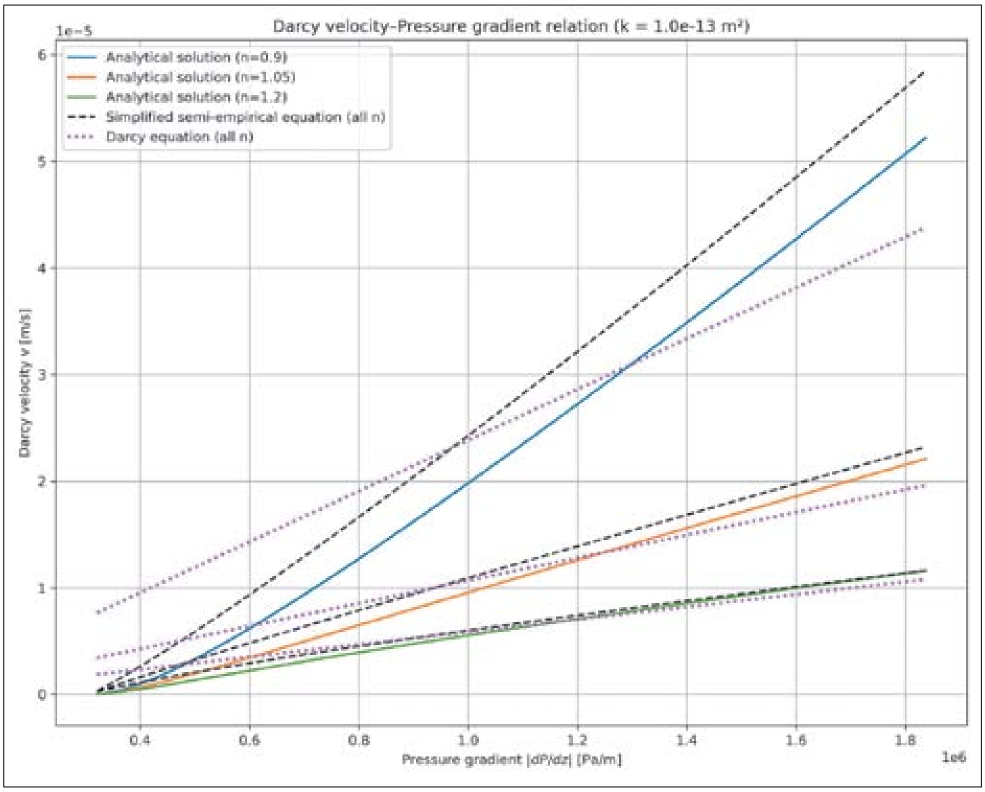

Figure 2. Relationship between velocity and the pressure gradient derived from equations (11) and (20) for various values of *n* when $k = 1.0\ e^{-13}\ m^2$

Rysunek 2. Zależność prędkości przepływu od gradientu ciśnienia wyprowadzona z równań (11) i (20) dla różnych wartości *n* przy $k = 1{,}0\ e^{-13}\ m^2$

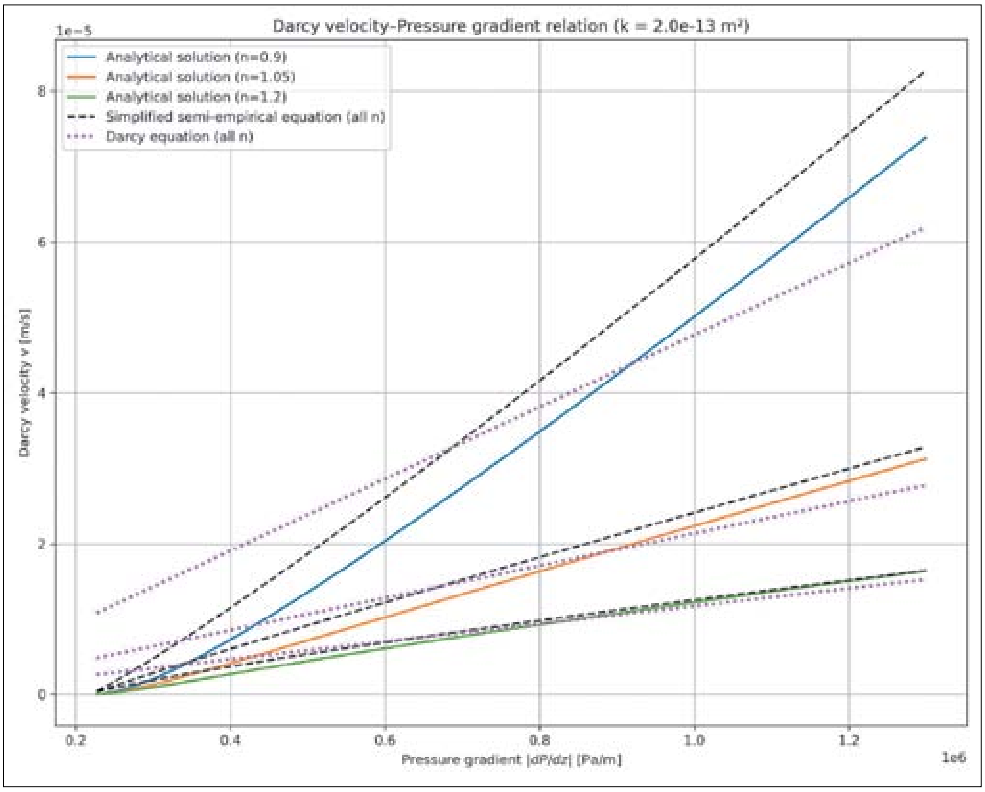

Figure 3. Relationship between velocity and the pressure gradient derived from equations (11) and (20) for various values of *n* when $k = 2.0\ e^{-13}\ m^2$

Rysunek 3. Zależność prędkości przepływu od gradientu ciśnienia wyprowadzona z równań (11) i (20) dla różnych wartości *n* przy $k = 2{,}0\ e^{-13}\ m^2$

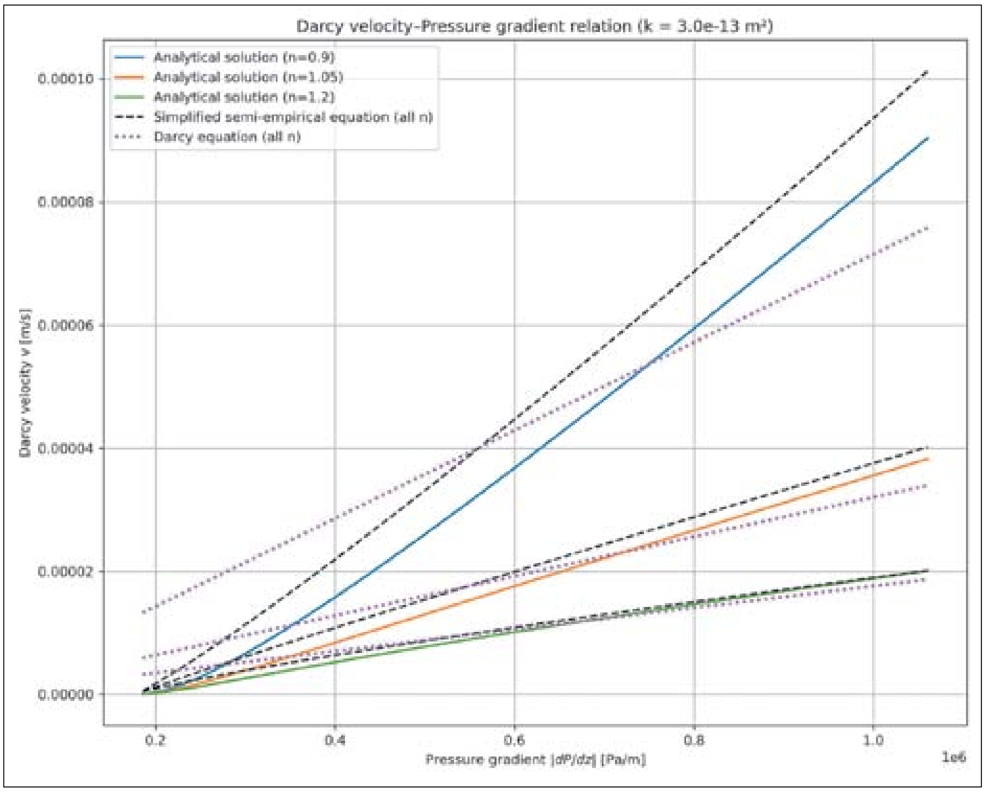

Figure 4. Relationship between velocity and the pressure gradient derived from equations (11) and (20) for various values of *n* when $k = 3.0\ e^{-13}\ m^2$

Rysunek 4. Zależność prędkości przepływu od gradientu ciśnienia wyprowadzona z równań (11) i (20) dla różnych wartości *n* przy $k = 3{,}0\ e^{-13}\ m^2$

relationship is very sensitive to changes in the value of *n*. Even a small change in *n* from 0.9 to 1.2 can decrease the velocity almost fivefold. Secondly, it can be seen how close the results obtained for the simplified equation (11) are to those for the more complex equation (20) that was derived in this work. The difference is insignificant, increasing only slightly for smaller values of *n*. Therefore, we conclude that the use of the simplified equation (11) is completely justified for any range of values of *n* and other parameters listed in Table 1. The purple dotted line shows the relationship between velocity and the pressure gradient, based on Darcy's law for the same permeability. The viscosity used in Darcy's law is found using the least squares method to obtain a value that minimizes the difference between the analytical solution and the Darcy equation. However, the Darcy equation remains inaccurate here as it fails to capture the trends relevant for a non-Newtonian fluid.

Figure 3 and Figure 4 show the relationship between velocity and the pressure gradient for various values of *n* for the analytical solution and the simplified semi-empirical equation for the values of permeability equal to $k = 2 \times 10^{-13}\ m^2$ and $k = 3 \times 10^{-13}\ m^2$ respectively. The analytical solution derived in this work and the simplified equation (11) yield similar results also for these permeabilities. The purple dotted line shows the relationship between velocity and the pressure gradient based on Darcy's equation, which yields inaccurate results in these cases as well, especially for lower values of *n*.

Table 2 shows the relationship between permeability and the initial pressure gradient derived from equation (26). The threefold increase in permeability decreases the value of the initial pressure gradient by approximately 1.7 times.

Table 2. Values of the initial pressure gradient at different values of permeability

Tabela 2. Wartości początkowego gradientu ciśnienia dla różnych wartości przepuszczalności

Permeability [m^2]	Initial pressure gradient [Pa/m]
1×10^{-13}	306200
2×10^{-13}	216500
3×10^{-13}	176800

Conclusions

This work derived the modified Blake-Kozeny equation for non-Newtonian Herschel-Bulkley fluids that flow in porous media, analogous to Darcy's law used for Newtonian fluids. The conclusion is that the presented equation for HB fluids is strongly non-linear. It is therefore complicated to use, especially in unsteady-state conditions, when pressure changes with time. It is doubtful whether the diffusivity equation that could be derived for this type of fluid could be solved. Relative to the pressure, the obtained equation is an ordinary differential equation, meaning that to calculate the pressure distribution in the porous media, we need to solve this equation for the pressure gradient, after which we can solve the ODE (ordinary differential equation). This must also be done to relate the flow rate to the boundary conditions (pressure values at the start and end of the porous media).

A comparative analysis of the fully analytical modified Blake-Kozeny equation for HB fluids derived in this work with the semi-empirical modified Darcy's law for HB fluids proposed in previous literature was conducted. The semi-empirical simplified equation yields results similar to the complex equation derived for the value ranges of the pressure gradient and n analyzed in this work. This means that the use of the simplified equation is completely justified. Therefore, we have analytically proved the validity of the simplified semi-empirical equation for accurately describing the behavior of all the mentioned non-Newtonian fluids in porous media. The key conclusion of this work is the derivation of a formula that allows the analytical determination of the minimum pressure gradient required for the flow of the fluid to start in porous media, instead of using an empirical coefficient β as proposed in previous literature. This is important because knowledge of the minimum pressure gradient is required to predict the formation of dead zones during reservoir modeling. In dead zones, this minimum gradient is not exceeded and therefore crude oil from these zones is not extracted. This allows to predict the final oil recovery factor more accurately.

This work analyzed three simplified cases of the HB fluid: the Darcy velocity equations for Newtonian fluids, Power Law fluids and Bingham Plastic fluids. The modified Blake-Kozeny equation for HB fluids is easily reduced to one of the three simplified models mentioned by setting the HB model parameters to their corresponding values. The PL fluid velocity equation derived in this work is the same as the equation previously derived in the presented literature. It is relatively simple and that's why it has been deeply studied. In contrast, the velocity equations derived for the Bingham Plastic fluids and Herschel-Bulkley fluids are much more complicated due to the presence of non-linear terms.

Nomenclature

v – velocity [m/s]
v_c – core velocity [m/s]
n – flow index [dimensionless]
z – coordinate along fluid flow [m]
r – radius [m]
r_H – hydraulic radius [m]
r_c – core radius [m]
k – permeability [m^2]
K – consistency index [Pa · s^n]
P – pressure [Pa]
C – tortuosity coefficient [dimensionless]
R – pipe radius [m]
V – total packed bed volume [m^3]
Q – total pipe flow rate [m^3/s]
Q_c – core flow rate [m^3/s]
Q_{oc} – out-of-core flow rate [m^3/s]
D_p – grain particle diameter [m]
R_p – grain particle radius [m]
A_p – grain particle surface area [m^2]
V_p – grain particle volume [m^3]
α_0 – initial pressure gradient [Pa/m]
$\dot{\gamma}$ – shear rate [s^{-1}]
τ – shear stress [Pa]
τ_0 – yield point [Pa]
ϕ – porosity [dimensionless]
μ – viscosity [Pa · s]
μ_B – Bingham plastic viscosity [Pa · s]
ρ – density [kg · m^{-3}]
μ_{ef} – effective viscosity [Pa · s]

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Appendix A

The Hagen–Poiseuille equation is (Kirby, 2010):

$$Q = (\pi R^4 \, \Delta P)/8\mu L \tag{A-1}$$

From equation (A-1) we conclude that the average velocity in a pipe is (dividing both sides by πR^2):

$$v = (R^2 \, \Delta P)/8\mu L \tag{A-2}$$

Since area of the pipe is πR^2 and perimeter of the pipe is $2\pi R$, we conclude that the hydraulic radius of a pipe is:

$$r_H = \pi R^2/2\pi R = R/2 \tag{A-3}$$

The average hydraulic radius of the porous media can be obtained by dividing the pore volume by the surface area of the grain particles:

$$r_H = (V - V_p)/A_p \tag{A-4}$$

In equation (A-4), V is the total volume, V_p is the particle volume (grain volume) and A_p is the particle surface area. The term $V - V_p$ stands for the pore volume. We know that the relation between particle volume and total volume is as follows:

$$V = V_p/(1 - \phi) \tag{A-5}$$

By inserting equation (A-5) into equation (A-4) we obtain:

$$r_H = V_p\phi/A_p(1 - \phi) \tag{A-6}$$

Now we consider that $A_p = 4\pi NR_p^2$, $V_p = 4\pi NR_p^3/3$ and substitute these definitions into equation (A-6).

Note that N stands for the number of particles. We obtain the following:

$$r_H = \frac{4\pi N R_p^3 \phi}{4\pi N R_p^2 * 3(1-\phi)} = \frac{R_p \phi}{3(1-\phi)} = \frac{D_p \phi}{6(1-\phi)} \quad \text{(A-7)}$$

Now we equate equation (A-7) with equation (A-3) and obtain that:

$$R = \frac{D_p \phi}{3(1-\phi)} \quad \text{(A-8)}$$

Now we substitute equation (A-8) into equation (A-2) and obtain:

$$v = \frac{D_p{}^2 \phi^2 \Delta P}{72(1-\phi)^2 \mu L} \quad \text{(A-9)}$$

Now we multiply equation (A-9) by ϕ to convert actual velocity to Darcy velocity and we also multiply it by 12/25 which is an empirical coefficient that is added to account for tortuosity (Christopher and Middleman, 1965). So, we obtain the Blake-Kozeny equation:

$$v = \frac{D_p{}^2 \phi^3 \Delta P}{150(1-\phi)^2 \mu L} \quad \text{(A-10)}$$

By equating the Blake-Kozeny equation to the Darcy law we obtain that the permeability is equal to:

$$k = \frac{D_p{}^2 \phi^3}{150(1-\phi)^2} \quad \text{(A-11)}$$

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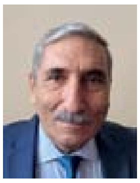

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